

# 7. Error Function and Fresnel Integrals

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## Contents

	Page
<b>Mathematical Properties</b> . . . . .	297
<b>7.1. Error Function</b> . . . . .	297
<b>7.2. Repeated Integrals of the Error Function</b> . . . . .	299
<b>7.3. Fresnel Integrals</b> . . . . .	300
<b>7.4. Definite and Indefinite Integrals</b> . . . . .	302
<b>Numerical Methods</b> . . . . .	304
<b>7.5. Use and Extension of the Tables</b> . . . . .	304
<b>References</b> . . . . .	308
<b>Table 7.1. Error Function and its Derivative (<math>0 \leq x \leq 2</math>)</b> . . . . .	310
$(2/\sqrt{\pi})e^{-x^2}, \operatorname{erf} x = (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt, x=0(.01)2, 10D$	
<b>Table 7.2. Derivative of the Error Function (<math>2 \leq x \leq 10</math>)</b> . . . . .	312
$(2/\sqrt{\pi})e^{-x^2}, x=2(.01)10, 8S$	
<b>Table 7.3. Complementary Error Function (<math>2 \leq x \leq \infty</math>)</b> . . . . .	316
$xe^{x^2} \operatorname{erfc} x = (2/\sqrt{\pi})xe^{x^2} \int_x^\infty e^{-t^2} dt, x^{-2}=.25(-.005)0, 7D$	
$\operatorname{erfc} \sqrt{n\pi}, n=1(1)10, 15D$	
<b>Table 7.4. Repeated Integrals of the Error Function (<math>0 \leq x \leq 5</math>)</b> . . . . .	317
$2^n \Gamma\left(\frac{n}{2}+1\right) i^n \operatorname{erfc} x = 2^{n+1} \Gamma\left(\frac{n}{2}+1\right) \frac{1}{\sqrt{\pi}} \int_x^\infty \frac{(t-x)^n}{n!} e^{-t^2} dt$	
$x=0(.1)5, n=1(1)6, 10, 11, 6S$	
<b>Table 7.5. Dawson's Integral (<math>0 \leq x \leq \infty</math>)</b> . . . . .	319
$e^{-x^2} \int_0^x e^{t^2} dt, x=0(.02)2, 10D$	
$xe^{-x^2} \int_0^x e^{t^2} dt, x^{-2}=.25(-.005)0, 9D$	

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	Page
<b>Table 7.6.</b> $(3/\Gamma(1/3)) \int_0^x e^{-t^3} dt$ ( $0 \leq x \leq 2.3$ ) . . . . .	320
$x=0(.02)1.7(.04)2.3$ , 7D	
<b>Table 7.7.</b> Fresnel Integrals ( $0 \leq x \leq 5$ ). . . . .	321
$C(x) = \int_0^x \cos\left(\frac{\pi}{2} t^2\right) dt$ , $S(x) = \int_0^x \sin\left(\frac{\pi}{2} t^2\right) dt$ , $x=0(.02)5$ , 7D	
<b>Table 7.8.</b> Auxiliary Functions ( $0 \leq x \leq \infty$ ) . . . . .	323
$f(x) = \left[\frac{1}{2} - S(x)\right] \cos\left(\frac{\pi}{2} x^2\right) - \left[\frac{1}{2} - C(x)\right] \sin\left(\frac{\pi}{2} x^2\right)$ $g(x) = \left[\frac{1}{2} - C(x)\right] \cos\left(\frac{\pi}{2} x^2\right) + \left[\frac{1}{2} - S(x)\right] \sin\left(\frac{\pi}{2} x^2\right)$ $x=0(.02)1$ , $x^{-1}=1(-.02)0$ , 15D	
<b>Table 7.9.</b> Error Function for Complex Arguments ( $0 \leq x \leq 3.9$ , $0 \leq y \leq 3$ ) . . . . .	325
$w(z) = e^{-z^2} \operatorname{erfc}(-iz)$ , $z = x + iy$ , $x=0(.1)3.9$ , $y=0(.1)3$ , 6D	
<b>Table 7.10.</b> Complex Zeros of the Error Function ( $1 \leq n \leq 10$ ) . . . . .	329
$z_n$ , $\operatorname{erf} z_n = 0$ , $n=1(1)10$ , 8D	
<b>Table 7.11.</b> Complex Zeros of Fresnel Integrals ( $0 \leq n \leq 5$ ) . . . . .	329
$z_n$ , $z_n^*$ , $C(z_n) = 0$ , $S(z_n^*) = 0$ , $n=0(1)5$ , 4D	
<b>Table 7.12.</b> Maxima and Minima of Fresnel Integrals ( $0 \leq n \leq 5$ ) . . . . .	329
$C(\sqrt{4n+1})$ , $C(\sqrt{4n+3})$ , $S(\sqrt{4n+2})$ , $S(\sqrt{4n+4})$ , $n=0(1)5$ , 6D	

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# 7. Error Function and Fresnel Integrals

## Mathematical Properties

### 7.1. Error Function

#### Definitions

$$7.1.1 \quad \operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

$$7.1.2 \quad \operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt = 1 - \operatorname{erf} z$$

$$7.1.3 \quad w(z) = e^{-z^2} \left( 1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{t^2} dt \right) = e^{-z^2} \operatorname{erfc}(-iz)$$

In 7.1.2 the path of integration is subject to the restriction  $\arg t \rightarrow \alpha$  with  $|\alpha| < \frac{\pi}{4}$  as  $t \rightarrow \infty$  along the path. ( $\alpha = \frac{\pi}{4}$  is permissible if  $\Re t^2$  remains bounded to the left.)

#### Integral Representation

$$7.1.4 \quad w(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{e^{-t^2} dt}{z-t} = \frac{2iz}{\pi} \int_0^{\infty} \frac{e^{-t^2} dt}{z^2 - t^2} \quad (\Im z > 0)$$

#### Series Expansions

$$7.1.5 \quad \operatorname{erf} z = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}$$

$$7.1.6 \quad = \frac{2}{\sqrt{\pi}} e^{-z^2} \sum_{n=0}^{\infty} \frac{2^n}{1 \cdot 3 \cdots (2n+1)} z^{2n+1}$$

$$7.1.7 \quad = \sqrt{2} \sum_{n=0}^{\infty} (-1)^n [I_{2n+1/2}(z^2) - I_{2n+3/2}(z^2)]$$

$$7.1.8 \quad w(z) = \sum_{n=0}^{\infty} \frac{(iz)^n}{\Gamma\left(\frac{n}{2} + 1\right)}$$

For  $I_{n-1/2}(x)$ , see chapter 10.

#### Symmetry Relations

$$7.1.9 \quad \operatorname{erf}(-z) = -\operatorname{erf} z$$

$$7.1.10 \quad \operatorname{erf} \bar{z} = \overline{\operatorname{erf} z}$$

$$7.1.11 \quad w(-z) = 2e^{-z^2} - w(z)$$

$$7.1.12 \quad w(\bar{z}) = \overline{w(-z)}$$

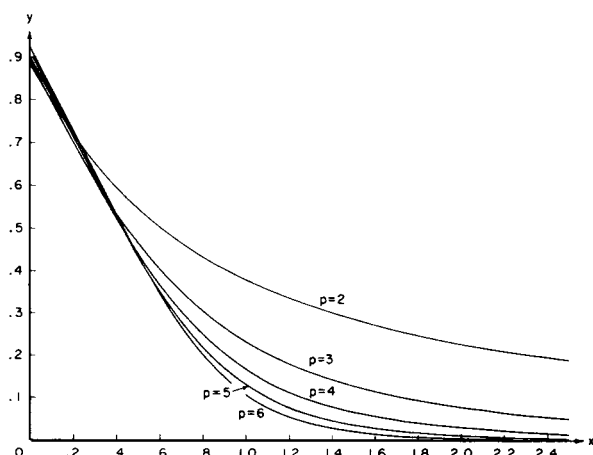


FIGURE 7.1.  $y = e^{-x^p} \int_x^\infty e^{-t^p} dt$ .  
 $p=2(1)6$

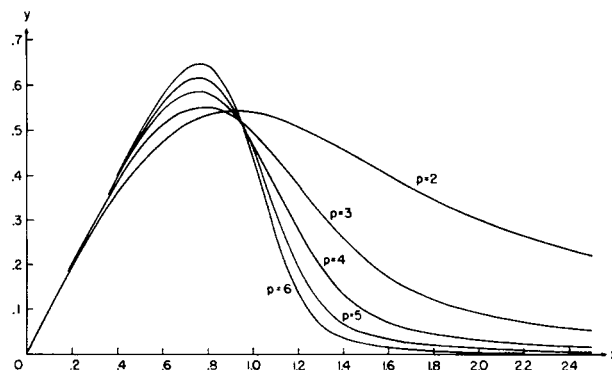
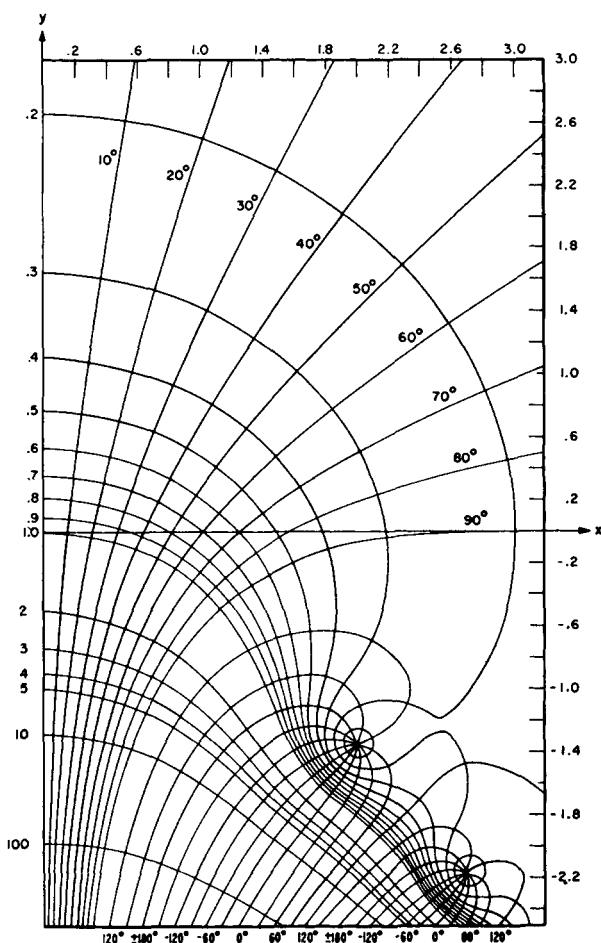


FIGURE 7.2.  $y = e^{-x^p} \int_0^x e^{t^p} dt$ .  
 $p=2(1)6$

FIGURE 7.3. Altitude Chart of  $w(z)$ .**Inequalities [7.11], [7.17]**

7.1.13

$$\frac{1}{x + \sqrt{x^2 + 2}} < e^{x^2} \int_x^\infty e^{-t^2} dt \leq \frac{1}{x + \sqrt{x^2 + \frac{4}{\pi}}} \quad (x \geq 0)$$

(For other inequalities see [7.2].)

**Continued Fractions**

7.1.14

$$2e^{x^2} \int_x^\infty e^{-t^2} dt = \frac{1}{z + \frac{1/2}{z + \frac{1}{z + \frac{3/2}{z + \frac{2}{z + \dots}}}}} \quad (\Re z > 0)$$

7.1.15

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{e^{-t^2} dt}{z - t} = \frac{1}{z - \frac{1/2}{z - \frac{1}{z - \frac{3/2}{z - \frac{2}{z - \dots}}}}} = \frac{1}{\sqrt{\pi}} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{H_k^{(n)}}{z - x_k^{(n)}} \quad (\Im z \neq 0)$$

$x_k^{(n)}$  and  $H_k^{(n)}$  are the zeros and weight factors of the Hermite polynomials. For numerical values see chapter 25.

**Value at Infinity**

$$7.1.16 \quad \operatorname{erf} z \rightarrow 1 \quad \left( z \rightarrow \infty \text{ in } |\arg z| < \frac{\pi}{4} \right)$$

**Maximum and Inflection Points for Dawson's Integral [7.31]**

$$F(x) = e^{-x^2} \int_0^x e^{t^2} dt$$

$$7.1.17 \quad F(.92413 \ 88730 \dots) = .54104 \ 42246 \dots$$

$$7.1.18 \quad F(1.50197 \ 52682 \dots) = .42768 \ 66160 \dots$$

**Derivatives**

7.1.19

$$\frac{d^{n+1}}{dz^{n+1}} \operatorname{erf} z = (-1)^n \frac{2}{\sqrt{\pi}} H_n(z) e^{-z^2} \quad (n=0, 1, 2, \dots)$$

7.1.20

$$w^{(n+2)}(z) + 2zw^{(n+1)}(z) + 2(n+1)w^{(n)}(z) = 0 \quad (n=0, 1, 2, \dots)$$

$$w^{(0)}(z) = w(z), \quad w'(z) = -2zw(z) + \frac{2i}{\sqrt{\pi}}$$

(For the Hermite polynomials  $H_n(z)$  see chapter 22.)

**Relation to Confluent Hypergeometric Function (see chapter 13)**

7.1.21

$$\operatorname{erf} z = \frac{2z}{\sqrt{\pi}} M\left(\frac{1}{2}, \frac{3}{2}, -z^2\right) = \frac{2z}{\sqrt{\pi}} e^{-z^2} M\left(1, \frac{3}{2}, z^2\right)$$

**The Normal Distribution Function With Mean  $m$  and Standard Deviation  $\sigma$  (see chapter 26)**

$$7.1.22 \quad \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(t-m)^2}{2\sigma^2}} dt = \frac{1}{2} \left( 1 + \operatorname{erf} \left( \frac{x-m}{\sigma\sqrt{2}} \right) \right)$$

**Asymptotic Expansion**

7.1.23

$$\sqrt{\pi} z e^{z^2} \operatorname{erfc} z \sim 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1 \cdot 3 \dots (2m-1)}{(2z^2)^m} \quad \left( z \rightarrow \infty, |\arg z| < \frac{3\pi}{4} \right)$$

If  $R_n(z)$  is the remainder after  $n$  terms then

## 7.1.24

$$R_n(z) = (-1)^n \frac{1 \cdot 3 \cdots (2n-1)}{(2z^2)^n} \theta,$$

$$\theta = \int_0^\infty e^{-t} \left(1 + \frac{t}{z^2}\right)^{-n-1} dt \quad \left(|\arg z| < \frac{\pi}{2}\right)$$

$$|\theta| < 1 \quad \left(|\arg z| < \frac{\pi}{4}\right)$$

For  $x$  real,  $R_n(x)$  is less in absolute value than the first neglected term and of the same sign.

Rational Approximations<sup>2</sup> ( $0 \leq x < \infty$ )

## 7.1.25

$$\operatorname{erf} x = 1 - (a_1 t + a_2 t^2 + a_3 t^3) e^{-x^2} + \epsilon(x), \quad t = \frac{1}{1+px}$$

$$|\epsilon(x)| \leq 2.5 \times 10^{-5}$$

$$p = .47047 \quad a_1 = .34802 \quad 42 \quad a_2 = -.09587 \quad 98$$

$$a_3 = .74785 \quad 56$$

## 7.1.26

$$\operatorname{erf} x = 1 - (a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5) e^{-x^2} + \epsilon(x),$$

$$t = \frac{1}{1+px}$$

$$|\epsilon(x)| \leq 1.5 \times 10^{-7}$$

$$p = .32759 \quad 11 \quad a_1 = .25482 \quad 9592$$

$$a_2 = -.28449 \quad 6736 \quad a_3 = 1.42141 \quad 3741$$

$$a_4 = -1.45315 \quad 2027 \quad a_5 = 1.06140 \quad 5429$$

## 7.1.27

$$\operatorname{erf} x = 1 - \frac{1}{[1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4]^4} + \epsilon(x)$$

$$|\epsilon(x)| \leq 5 \times 10^{-4}$$

$$a_1 = .278393 \quad a_2 = .230389$$

$$a_3 = .000972 \quad a_4 = .078108$$

## 7.1.28

$$\operatorname{erf} x = 1 - \frac{1}{[1 + a_1 x + a_2 x^2 + \cdots + a_6 x^6]^{16}} + \epsilon(x)$$

$$|\epsilon(x)| \leq 3 \times 10^{-7}$$

$$a_1 = .07052 \quad 30784 \quad a_2 = .04228 \quad 20123$$

$$a_3 = .00927 \quad 05272 \quad a_4 = .00015 \quad 20143$$

$$a_5 = .00027 \quad 65672 \quad a_6 = .00004 \quad 30638$$

<sup>2</sup> Approximations 7.1.25-7.1.28 are from C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N. J., 1955 (with permission).

## Infinite Series Approximation for Complex Error Function [7.19]

## 7.1.29

$$\operatorname{erf}(x+iy) = \operatorname{erf} x + \frac{e^{-x^2}}{2\pi x} [(1 - \cos 2xy) + i \sin 2xy]$$

$$+ \frac{2}{\pi} e^{-x^2} \sum_{n=1}^{\infty} \frac{e^{-4n^2}}{n^2 + 4x^2} [f_n(x, y) + i g_n(x, y)] + \epsilon(x, y)$$

where

$$f_n(x, y) = 2x - 2x \cosh ny \cos 2xy + n \sinh ny \sin 2xy$$

$$g_n(x, y) = 2x \cosh ny \sin 2xy + n \sinh ny \cos 2xy$$

$$|\epsilon(x, y)| \approx 10^{-16} |\operatorname{erf}(x+iy)|$$

## 7.2. Repeated Integrals of the Error Function

## Definition

## 7.2.1

$$i^n \operatorname{erfc} z = \int_z^\infty i^{n-1} \operatorname{erfc} t \, dt \quad (n=0, 1, 2, \dots)$$

$$i^{-1} \operatorname{erfc} z = \frac{2}{\sqrt{\pi}} e^{-z^2}, \quad i^0 \operatorname{erfc} z = \operatorname{erfc} z$$

## Differential Equation

$$7.2.2 \quad \frac{d^2 y}{dz^2} + 2z \frac{dy}{dz} - 2ny = 0$$

$$y = A i^n \operatorname{erfc} z + B i^n \operatorname{erfc}(-z)$$

( $A$  and  $B$  are constants.)

## Expression as a Single Integral

$$7.2.3 \quad i^n \operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^\infty \frac{(t-z)^n}{n!} e^{-t^2} dt$$

Power Series<sup>3</sup>

$$7.2.4 \quad i^n \operatorname{erfc} z = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{2^{n-k} k! \Gamma\left(1 + \frac{n-k}{2}\right)}$$

## Recurrence Relations

## 7.2.5

$$i^n \operatorname{erfc} z = -\frac{z}{n} i^{n-1} \operatorname{erfc} z + \frac{1}{2n} i^{n-2} \operatorname{erfc} z$$

$$(n=1, 2, 3, \dots)$$

## 7.2.6

$$2(n+1)(n+2) i^{n+2} \operatorname{erfc} z$$

$$= (2n+1+2z^2) i^n \operatorname{erfc} z - \frac{1}{2} i^{n-2} \operatorname{erfc} z$$

$$(n=1, 2, 3, \dots)$$

<sup>3</sup> The terms in this series corresponding to  $k=n+2, n+4, n+6, \dots$  are understood to be zero.

## 7.2.7

## Value at Zero

$$i^n \operatorname{erfc} 0 = \frac{1}{2^n \Gamma\left(\frac{n}{2} + 1\right)} \quad (n = -1, 0, 1, 2, \dots)$$

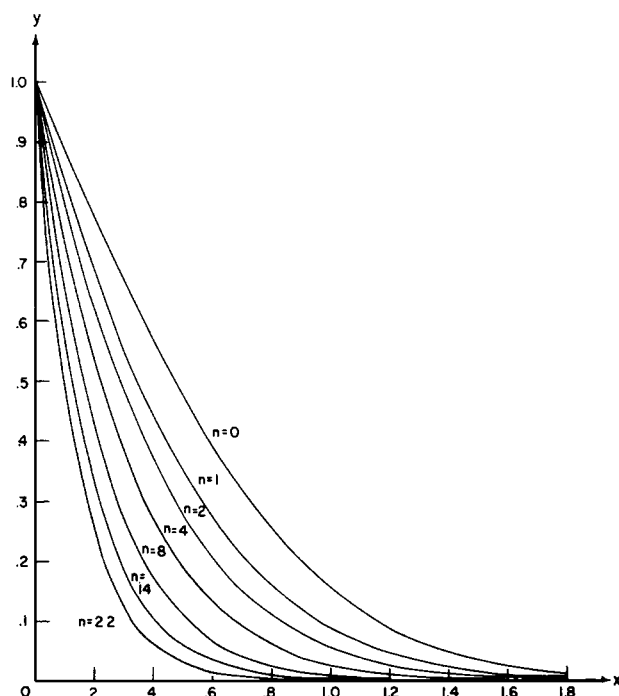


FIGURE 7.4. Repeated Integrals of the Error Function.

$$y = 2^n \Gamma\left(\frac{n}{2} + 1\right) i^n \operatorname{erfc} z$$

$n = 0, 1, 2, 4, 8, 14, 22$

## Derivatives

$$7.2.8 \quad \frac{d}{dz} i^n \operatorname{erfc} z = -i^{n-1} \operatorname{erfc} z \quad (n = 0, 1, 2, \dots)$$

## 7.2.9

$$\frac{d^n}{dz^n} (e^{z^2} \operatorname{erfc} z) = (-1)^n 2^n n! e^{z^2} i^n \operatorname{erfc} z$$

$(n = 0, 1, 2, \dots)$

Relation to  $Hh_n(z)$  (see 19.14)

$$7.2.10 \quad i^n \operatorname{erfc} z = \frac{1}{(2^{n-1} \pi)^{\frac{1}{2}}} Hh_n(\sqrt{2} z)$$

## Relation to Hermite Polynomials (see chapter 22)

$$7.2.11 \quad (-1)^n i^n \operatorname{erfc} z + i^n \operatorname{erfc} (-z) = \frac{i^{-n}}{2^{n-1} n!} H_n(iz)$$

Relation to the Confluent Hypergeometric Function  
(see chapter 13)

## 7.2.12

$$i^n \operatorname{erfc} z = e^{-z^2} \left[ \frac{1}{2^n \Gamma\left(\frac{n}{2} + 1\right)} M\left(\frac{n+1}{2}, \frac{1}{2}, z^2\right) - \frac{z}{2^{n-1} \Gamma\left(\frac{n+1}{2}\right)} M\left(\frac{n}{2} + 1, \frac{3}{2}, z^2\right) \right]$$

## Relation to Parabolic Cylinder Functions (see chapter 19)

$$7.2.13 \quad i^n \operatorname{erfc} z = \frac{e^{-\frac{1}{2}z^2}}{(2^{n-1} \pi)^{\frac{1}{2}}} D_{-n-1}(z\sqrt{2})$$

## Asymptotic Expansion

## 7.2.14

$$i^n \operatorname{erfc} z \sim \frac{2}{\sqrt{\pi}} \frac{e^{-z^2}}{(2z)^{n+1}} \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)!}{n! m! (2z)^{2m}}$$

$(z \rightarrow \infty, |\arg z| < \frac{3\pi}{4})$

## 7.3. Fresnel Integrals

## Definition

$$7.3.1 \quad C(z) = \int_0^z \cos\left(\frac{\pi}{2} t^2\right) dt$$

$$7.3.2 \quad S(z) = \int_0^z \sin\left(\frac{\pi}{2} t^2\right) dt$$

The following functions are also in use:

## 7.3.3

$$C_1(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos t^2 dt, \quad C_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}} dt$$

## 7.3.4

$$S_1(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin t^2 dt, \quad S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}} dt$$

## Auxiliary Functions

## 7.3.5

$$f(z) = \left[\frac{1}{2} - S(z)\right] \cos\left(\frac{\pi}{2} z^2\right) - \left[\frac{1}{2} - C(z)\right] \sin\left(\frac{\pi}{2} z^2\right)$$

## 7.3.6

$$g(z) = \left[\frac{1}{2} - C(z)\right] \cos\left(\frac{\pi}{2} z^2\right) + \left[\frac{1}{2} - S(z)\right] \sin\left(\frac{\pi}{2} z^2\right)$$

## Interrelations

$$7.3.7 \quad C(x) = C_1\left(x\sqrt{\frac{\pi}{2}}\right) = C_2\left(\frac{\pi}{2} x^2\right)$$

$$7.3.8 \quad S(x) = S_1\left(x\sqrt{\frac{\pi}{2}}\right) = S_2\left(\frac{\pi}{2}x^2\right)$$

$$7.3.9 \quad C(z) = \frac{1}{2} + f(z) \sin\left(\frac{\pi}{2}z^2\right) - g(z) \cos\left(\frac{\pi}{2}z^2\right)$$

$$7.3.10 \quad S(z) = \frac{1}{2} - f(z) \cos\left(\frac{\pi}{2}z^2\right) - g(z) \sin\left(\frac{\pi}{2}z^2\right)$$

#### Series Expansions

$$7.3.11 \quad C(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n}}{(2n)!(4n+1)} z^{4n+1}$$

7.3.12

$$C(z) = \cos\left(\frac{\pi}{2}z^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{1 \cdot 3 \cdots (4n+1)} z^{4n+1} \\ + \sin\left(\frac{\pi}{2}z^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{1 \cdot 3 \cdots (4n+3)} z^{4n+3}$$

$$7.3.13 \quad S(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n+1}}{(2n+1)!(4n+3)} z^{4n+3}$$

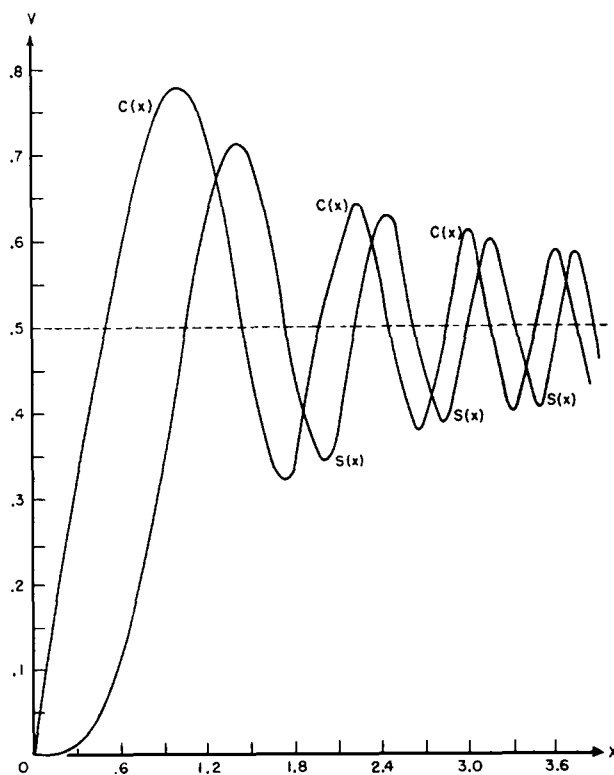


FIGURE 7.5. *Fresnel Integrals.*  
 $y = C(x), y = S(x)$

7.3.14

$$S(z) = -\cos\left(\frac{\pi}{2}z^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{1 \cdot 3 \cdots (4n+3)} z^{4n+3} \\ + \sin\left(\frac{\pi}{2}z^2\right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{1 \cdot 3 \cdots (4n+1)} z^{4n+1}$$

$$7.3.15 \quad C_2(z) = J_{1/2}(z) + J_{5/2}(z) + J_{9/2}(z) + \cdots$$

$$7.3.16 \quad S_2(z) = J_{3/2}(z) + J_{7/2}(z) + J_{11/2}(z) + \cdots$$

For Bessel functions  $J_{n+1/2}(z)$  see chapter 10.

#### Symmetry Relations

$$7.3.17 \quad C(-z) = -C(z), \quad S(-z) = -S(z)$$

$$7.3.18 \quad C(iz) = iC(z), \quad S(iz) = -iS(z)$$

$$7.3.19 \quad C(\bar{z}) = \overline{C(z)}, \quad S(\bar{z}) = \overline{S(z)}$$

#### Value at Infinity

$$7.3.20 \quad C(x) \rightarrow \frac{1}{2}, \quad S(x) \rightarrow \frac{1}{2} \quad (x \rightarrow \infty)$$

#### Derivatives

$$7.3.21 \quad \frac{dC(x)}{dx} = -\pi x g(x), \quad \frac{dS(x)}{dx} = \pi x f(x) - 1$$

#### Relation to Error Function (see 7.1.1, 7.1.3)

7.3.22

$$C(z) + iS(z) = \frac{1+i}{2} \operatorname{erf}\left[\frac{\sqrt{\pi}}{2}(1-i)z\right] \\ = \frac{1+i}{2} \left\{ 1 - e^{i\frac{\pi}{2}z^2} w\left[\frac{\sqrt{\pi}}{2}(1+i)z\right] \right\}$$

$$7.3.23 \quad g(x) = \mathcal{R}\left\{\frac{1+i}{2} w\left[\frac{\sqrt{\pi}}{2}(1+i)x\right]\right\}$$

$$7.3.24 \quad f(x) = \mathcal{I}\left\{\frac{1+i}{2} w\left[\frac{\sqrt{\pi}}{2}(1+i)x\right]\right\}$$

#### Relation to Confluent Hypergeometric Function (see chapter 13)

7.3.25

$$C(z) + iS(z) = zM\left(\frac{1}{2}, \frac{3}{2}, i\frac{\pi}{2}z^2\right) \\ = ze^{i\frac{\pi}{2}z^2} M\left(1, \frac{3}{2}, -i\frac{\pi}{2}z^2\right)$$

#### Relation to Spherical Bessel Functions (see chapter 10)

$$7.3.26 \quad C_2(z) = \frac{1}{2} \int_0^z J_{-1/2}(t) dt, \quad S_2(z) = \frac{1}{2} \int_0^z J_{1/2}(t) dt$$

## Asymptotic Expansions

## 7.3.27

$$\pi z f(z) \sim 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1 \cdot 3 \dots (4m-1)}{(\pi z^2)^{2m}} \quad \left( z \rightarrow \infty, |\arg z| < \frac{\pi}{2} \right)$$

## 7.3.28

$$\pi z g(z) \sim \sum_{m=0}^{\infty} (-1)^m \frac{1 \cdot 3 \dots (4m+1)}{(\pi z^2)^{2m+1}} \quad \left( z \rightarrow \infty, |\arg z| < \frac{\pi}{2} \right)$$

If  $R_n^{(f)}(z)$ ,  $R_n^{(g)}(z)$  are the remainders after  $n$  terms in 7.3.27, 7.3.28, respectively, then

## 7.3.29

$$R_n^{(f)}(z) = (-1)^n \frac{1 \cdot 3 \dots (4n-1)}{(\pi z^2)^{2n}} \theta^{(f)},$$

$$\theta^{(f)} = \frac{1}{\Gamma(2n + \frac{1}{2})} \int_0^{\infty} \frac{e^{-t} t^{2n-\frac{1}{2}}}{1 + \left(\frac{2t}{\pi z^2}\right)^2} dt \quad \left( |\arg z| < \frac{\pi}{4} \right)$$

## 7.3.30

$$R_n^{(g)}(z) = (-1)^n \frac{1 \cdot 3 \dots (4n+1)}{(\pi z^2)^{2n}} \theta^{(g)},$$

$$\theta^{(g)} = \frac{1}{\Gamma(2n + \frac{3}{2})} \int_0^{\infty} \frac{e^{-t} t^{2n+\frac{1}{2}}}{1 + \left(\frac{2t}{\pi z^2}\right)^2} dt \quad \left( |\arg z| < \frac{\pi}{4} \right)$$

$$7.3.31 \quad |\theta^{(f)}| < 1, |\theta^{(g)}| < 1 \quad \left( |\arg z| \leq \frac{\pi}{8} \right)$$

For  $x$  real,  $R_n^{(f)}(x)$  and  $R_n^{(g)}(x)$  are less in absolute value than the first neglected term and of the same sign.

Rational Approximations<sup>4</sup> ( $0 \leq x \leq \infty$ )

## 7.3.32

$$f(x) = \frac{1 + .926x}{2 + 1.792x + 3.104x^2} + \epsilon(x) \quad |\epsilon(x)| \leq 2 \times 10^{-3}$$

## 7.3.33

$$g(x) = \frac{1}{2 + 4.142x + 3.492x^2 + 6.670x^3} + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-3}$$

(For more accurate approximations see [7.1].)

## 7.4. Definite and Indefinite Integrals

For a more extensive list of integrals see [7.5], [7.8], [7.15].

## 7.4.1

$$\int_0^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

## 7.4.2

$$\int_0^{\infty} e^{-(at^2+2bt+c)} dt = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{\frac{b^2-ac}{a}} \operatorname{erfc} \frac{b}{\sqrt{a}} \quad (\Re a > 0)$$

## 7.4.3

$$\int_0^{\infty} e^{-at^2 - \frac{b}{t^2}} dt = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \quad (\Re a > 0, \Re b > 0)$$

## 7.4.4

$$\int_0^{\infty} t^{2n} e^{-at^2} dt = \frac{1 \cdot 3 \dots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}$$

$$= \frac{\Gamma(n + \frac{1}{2})}{2a^{n+\frac{1}{2}}} \quad (\Re a > 0; n=0, 1, 2, \dots)$$

## 7.4.5

$$\int_0^{\infty} t^{2n+1} e^{-at^2} dt = \frac{n!}{2a^{n+1}} \quad (\Re a > 0; n=0, 1, 2, \dots)$$

## 7.4.6

$$\int_0^{\infty} e^{-at^2} \cos(2xt) dt = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-\frac{x^2}{a}} \quad (\Re a > 0)$$

## 7.4.7

$$\int_0^{\infty} e^{-at^2} \sin(2xt) dt = \frac{1}{\sqrt{a}} e^{-x^2/a} \int_0^{x/\sqrt{a}} e^{t^2} dt$$

$$(\Re a > 0)$$

## 7.4.8

$$\int_0^{\infty} \frac{e^{-at} dt}{\sqrt{t+z^2}} = \sqrt{\frac{\pi}{a}} e^{az^2} \operatorname{erfc} \sqrt{az} \quad (\Re a > 0, \Re z > 0)$$

## 7.4.9

$$\int_0^{\infty} \frac{e^{-at} dt}{\sqrt{t(t+z)}} = \frac{\pi}{\sqrt{z}} e^{az} \operatorname{erfc} \sqrt{az}$$

$$(\Re a > 0, z \neq 0, |\arg z| < \pi)$$

## 7.4.10

$$\int_0^{\infty} \frac{e^{-at^2} dt}{t+x} = e^{-ax^2} \left[ \sqrt{\pi} \int_0^{\sqrt{ax}} e^{t^2} dt - \frac{1}{2} \operatorname{Ei}(ax^2) \right] \quad *$$

$$(a > 0, x > 0)$$

## 7.4.11

$$\int_0^{\infty} \frac{e^{-at^2} dt}{t^2+x^2} = \frac{\pi}{2x} e^{ax^2} \operatorname{erfc} \sqrt{ax} \quad (a > 0, x > 0)$$

$$7.4.12 \quad \int_0^1 \frac{e^{-at^2} dt}{t^2+1} = \frac{\pi}{4} e^a [1 - (\operatorname{erf} \sqrt{a})^2] \quad (a > 0)$$

## 7.4.13

$$\int_{-\infty}^{\infty} \frac{ye^{-t^2} dt}{(x-t)^2+y^2} = \pi \mathcal{R}w(x+iy) \quad (x \text{ real}, y > 0)$$

<sup>4</sup> Approximations 7.3.32, 7.3.33 are based on those given in C. Hastings, Jr., Approximations for calculating Fresnel integrals, Approximation Newsletter, April 1956, Note 10. [See also MTAC 10, 173, 1956.]



7.4.14

$$\int_{-\infty}^{\infty} \frac{(x-t)e^{-t^2} dt}{(x-t)^2+y^2} = \pi \mathcal{J} w(x+iy) \quad (x \text{ real}, y > 0)$$

7.4.15

$$\int_0^{\infty} \frac{[t^2 - (x^2 - y^2)]e^{-t^2} dt}{t^4 - 2(x^2 - y^2)t^2 + (x^2 + y^2)^2} = \frac{\pi}{2} \mathcal{R} \frac{w(x+iy)}{y-ix} \\ (x \text{ real}, y > 0)$$

7.4.16

$$\int_0^{\infty} \frac{2xye^{-t^2} dt}{t^4 - 2(x^2 - y^2)t^2 + (x^2 + y^2)^2} = \frac{\pi}{2} \mathcal{J} \frac{w(x+iy)}{y-ix} \\ (x \text{ real}, y > 0)$$

7.4.17

$$\int_0^{\infty} e^{-at} \operatorname{erf} bt \, dt = \frac{1}{a} e^{\frac{a^2}{4b^2}} \operatorname{erfc} \frac{a}{2b} \\ (\mathcal{R}a > 0, |\arg b| < \frac{\pi}{4})$$

7.4.18

$$\int_0^{\infty} \sin(2at) \operatorname{erfc} bt \, dt = \frac{1}{2a} [1 - e^{-(a/b)^2}] (a > 0, \mathcal{R}b > 0)$$

7.4.19

$$\int_0^{\infty} e^{-at} \operatorname{erf} \sqrt{bt} \, dt = \frac{1}{a} \sqrt{\frac{b}{a+b}} \quad (\mathcal{R}(a+b) > 0)$$

7.4.20

$$\int_0^{\infty} e^{-at} \operatorname{erfc} \sqrt{\frac{b}{t}} \, dt = \frac{1}{a} e^{-2\sqrt{ab}} \quad (\mathcal{R}a > 0, \mathcal{R}b > 0)$$

7.4.21

$$\int_0^{\infty} e^{(a-b)t} \operatorname{erfc} \left( \sqrt{at} + \sqrt{\frac{c}{t}} \right) dt = \frac{e^{-2(\sqrt{ac} + \sqrt{bc})}}{\sqrt{b}(\sqrt{a} + \sqrt{b})} \\ (\mathcal{R}b > 0, \mathcal{R}c > 0)$$

7.4.22

$$\int_0^{\infty} e^{-at} \cos(t^2) dt = \sqrt{\frac{\pi}{2}} \left\{ \left[ \frac{1}{2} - S\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right) \right] \cos\left(\frac{a^2}{4}\right) - \left[ \frac{1}{2} - C\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right) \right] \sin\left(\frac{a^2}{4}\right) \right\} \quad (\mathcal{R}a > 0)$$

7.4.23

$$\int_0^{\infty} e^{-at} \sin(t^2) dt = \sqrt{\frac{\pi}{2}} \left\{ \left[ \frac{1}{2} - C\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right) \right] \cos\left(\frac{a^2}{4}\right) + \left[ \frac{1}{2} - S\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right) \right] \sin\left(\frac{a^2}{4}\right) \right\} \quad (\mathcal{R}a > 0)$$

7.4.24

$$\int_0^{\infty} e^{-at} \frac{\sin(t^2)}{t} dt = \frac{\pi}{2} \left[ \frac{1}{2} - C\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right) \right]^2 \\ + \frac{\pi}{2} \left[ \frac{1}{2} - S\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right) \right]^2 \quad (\mathcal{R}a > 0)$$

7.4.25

$$\int_0^{\infty} \frac{e^{-at}\sqrt{t}}{t^2+b^2} dt = \pi \sqrt{\frac{2}{b}} \left\{ \left[ \frac{1}{2} - C\left(\sqrt{\frac{2ab}{\pi}}\right) \right] \cos(ab) + \left[ \frac{1}{2} - S\left(\sqrt{\frac{2ab}{\pi}}\right) \right] \sin(ab) \right\} \quad (\mathcal{R}a > 0, \mathcal{R}b > 0)$$

7.4.26

$$\int_0^{\infty} \frac{e^{-at} dt}{\sqrt{t}(t^2+b^2)} = \frac{\pi}{b} \sqrt{\frac{2}{b}} \left\{ \left[ \frac{1}{2} - S\left(\sqrt{\frac{2ab}{\pi}}\right) \right] \cos(ab) - \left[ \frac{1}{2} - C\left(\sqrt{\frac{2ab}{\pi}}\right) \right] \sin(ab) \right\} \quad (\mathcal{R}a > 0, \mathcal{R}b > 0)$$

7.4.27

$$\int_0^{\infty} e^{-at} C(t) dt = \frac{1}{a} \left\{ \left[ \frac{1}{2} - S\left(\frac{a}{\pi}\right) \right] \cos\left(\frac{a^2}{2\pi}\right) - \left[ \frac{1}{2} - C\left(\frac{a}{\pi}\right) \right] \sin\left(\frac{a^2}{2\pi}\right) \right\} \quad (\mathcal{R}a > 0)$$

7.4.28

$$\int_0^{\infty} e^{-at} S(t) dt = \frac{1}{a} \left\{ \left[ \frac{1}{2} - C\left(\frac{a}{\pi}\right) \right] \cos\left(\frac{a^2}{2\pi}\right) + \left[ \frac{1}{2} - S\left(\frac{a}{\pi}\right) \right] \sin\left(\frac{a^2}{2\pi}\right) \right\} \quad (\mathcal{R}a > 0)$$

7.4.29

$$\int_0^{\infty} e^{-at} C\left(\sqrt{\frac{2t}{\pi}}\right) dt = \frac{1}{2a(\sqrt{a^2+1}-a)\sqrt{a^2+1}} \quad (\mathcal{R}a > 0)$$

7.4.30

$$\int_0^{\infty} e^{-at} S\left(\sqrt{\frac{2t}{\pi}}\right) dt = \frac{1}{2a(\sqrt{a^2+1}+a)\sqrt{a^2+1}} \quad (\mathcal{R}a > 0)$$

$$7.4.31 \quad \int_0^{\infty} \left\{ \left[ \frac{1}{2} - C(t) \right]^2 + \left[ \frac{1}{2} - S(t) \right]^2 \right\} dt = \frac{1}{\pi}$$

7.4.32

$$\int e^{-(ax^2+2bx+c)} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{\frac{b^2-ac}{a}} \operatorname{erf}\left(\sqrt{a}x + \frac{b}{\sqrt{a}}\right) + \text{const.} \quad (a \neq 0)$$

## 7.4.33

$$\int e^{-a^2x^2 - \frac{b^2}{x^2}} dx = \frac{\sqrt{\pi}}{4a} \left[ e^{2ab} \operatorname{erf} \left( ax + \frac{b}{x} \right) + e^{-2ab} \operatorname{erf} \left( ax - \frac{b}{x} \right) \right] + \text{const.} \quad (a \neq 0)$$

## 7.4.34

$$\int e^{-a^2x^2 + \frac{b^2}{x^2}} dx = -\frac{\sqrt{\pi}}{4a} e^{-a^2x^2 + \frac{b^2}{x^2}} \left[ w \left( \frac{b}{x} + iax \right) + w \left( -\frac{b}{x} + iax \right) \right] + \text{const.} \quad (a \neq 0)$$

$$7.4.35 \quad \int \operatorname{erf} x dx = x \operatorname{erf} x + \frac{1}{\sqrt{\pi}} e^{-x^2} + \text{const.}$$

## 7.4.36

$$\int e^{ax} \operatorname{erf} bx dx = \frac{1}{a} \left[ e^{ax} \operatorname{erf} bx - e^{\frac{a^2}{4b^2}} \operatorname{erf} \left( bx - \frac{a}{2b} \right) \right] + \text{const.} \quad (a \neq 0)$$

## 7.4.37

$$\int e^{ax} \operatorname{erf} \sqrt{\frac{b}{x}} dx = \frac{1}{a} \left\{ e^{ax} \operatorname{erf} \sqrt{\frac{b}{x}} + \frac{1}{2} e^{ax - \frac{b}{x}} \left[ w \left( \sqrt{ax} + i \sqrt{\frac{b}{x}} \right) + w \left( -\sqrt{ax} + i \sqrt{\frac{b}{x}} \right) \right] \right\} + \text{const.} \quad (a \neq 0)$$

## 7.4.38

$$\begin{aligned} \int \cos(ax^2 + 2bx + c) dx \\ = \sqrt{\frac{\pi}{2a}} \left\{ \cos \left( \frac{b^2 - ac}{a} \right) C \left[ \sqrt{\frac{2}{a\pi}} (ax + b) \right] + \sin \left( \frac{b^2 - ac}{a} \right) S \left[ \sqrt{\frac{2}{a\pi}} (ax + b) \right] \right\} + \text{const.} \end{aligned}$$

## 7.4.39

$$\begin{aligned} \int \sin(ax^2 + 2bx + c) dx \\ = \sqrt{\frac{\pi}{2a}} \left\{ \cos \left( \frac{b^2 - ac}{a} \right) S \left[ \sqrt{\frac{2}{a\pi}} (ax + b) \right] - \sin \left( \frac{b^2 - ac}{a} \right) C \left[ \sqrt{\frac{2}{a\pi}} (ax + b) \right] \right\} + \text{const.} \end{aligned}$$

$$7.4.40 \quad \int C(x) dx = xC(x) - \frac{1}{\pi} \sin \left( \frac{\pi}{2} x^2 \right) + \text{const.}$$

$$7.4.41 \quad \int S(x) dx = xS(x) + \frac{1}{\pi} \cos \left( \frac{\pi}{2} x^2 \right) + \text{const.}$$

## Numerical Methods

## 7.5. Use and Extension of the Tables

**Example 1.** Compute  $\operatorname{erf} .745$  and  $e^{-(.745)^2}$  using Taylor's series.

With the aid of Taylor's theorem and 7.1.19 it can be shown that

$$\operatorname{erf} (x_0 + ph) = \operatorname{erf} x_0 + \frac{2}{\sqrt{\pi}} e^{-x_0^2} ph \left[ 1 - phx_0 + \frac{1}{3} p^2 h^2 (2x_0^2 - 1) \right] + \epsilon$$

$$e^{-(x_0 + ph)^2} = e^{-x_0^2} \left[ 1 - 2phx_0 + p^2 h^2 (2x_0^2 - 1) - \frac{2}{3} p^3 h^3 x_0 (2x_0^2 - 3) \right] + \eta$$

where  $|\epsilon| < 1.2 \times 10^{-10}$ ,  $|\eta| < 3.2 \times 10^{-10}$  if  $h = 10^{-2}$ ,  $|p| \leq \frac{1}{2}$ . With  $x_0 = .74$ ,  $p = .5$  and using Table 7.1

$$\begin{aligned} \operatorname{erf} .745 &= .70467 \ 80779 + (.5)(.00652 \ 58247) \times \\ &\quad [1 - (.005)(.74) + (.00000 \ 83333)(.0952)] \\ &= .70792 \ 8920 \end{aligned}$$

$$\begin{aligned} e^{-(.745)^2} &= \frac{\sqrt{\pi}}{2} (.65258 \ 24665) [1 - .0074 \\ &\quad + (.000025)(.0952) + (.00000 \ 00833)(.74)(1.9048)] \\ &= .57405 \ 7910. \end{aligned}$$

As a check the computation was repeated with  $x_0 = .75$ ,  $p = -.5$ .

**Example 2.** Compute  $\operatorname{erfc} x$  to 5S for  $x = 4.8$ . We have  $1/x^2 = .0434028$ . With Table 7.2 and linear interpolation in Table 7.3, we obtain

$$\begin{aligned} \operatorname{erfc} 4.8 &= \frac{1}{4.8} (1.11253)(10^{-10})(.552669) \frac{\sqrt{\pi}}{2} \\ &= (1.1352)10^{-11}. \end{aligned}$$

**Example 3.** Compute  $e^{-x^2} \int_0^x e^{t^2} dt$  to 5S for  $x=6.5$ .

With  $1/x^2=.0236686$  and linear interpolation in **Table 7.5**

$$e^{-(6.5)^2} \int_0^{6.5} e^{t^2} dt = (.506143)/(6.5) = .077868.$$

**Example 4.** Compute  $i^2 \operatorname{erfc} 1.72$  using the recurrence relation and **Table 7.1**.

By 7.2.1, using **Table 7.1**,

$$i^{-1} \operatorname{erfc} 1.72 = .05856 \ 50.$$

Using the recurrence relation 7.2.5 and **Table 7.1**

$$i \operatorname{erfc} 1.72 = -(1.72)(.01499 \ 72) + (.5)(.05856 \ 50) \\ = .0034873$$

$$i^2 \operatorname{erfc} 1.72 = -(.86)(.0034873) + (.25)(.01499 \ 72) \\ = .0007502.$$

Note the loss of two significant digits.

**Example 5.** Compute  $i^k \operatorname{erfc} 1.72$  for  $k=1, 2, 3$  by backward recurrence.

Let the sequence  $w_\mu^m(x)$  ( $\mu=m, m-1, \dots, 1, 0, -1$ ) be generated by backward use of the recurrence relation 7.2.5 starting with  $w_{m+2}^m=0$ ,  $w_{m+1}^m=1$ . Then, for any fixed  $k$ , (see [7.7]),

$$\lim_{m \rightarrow \infty} \frac{w_k^m(x)}{w_{-1}^m(x)} = \frac{\sqrt{\pi}}{2} e^{x^2} i^k \operatorname{erfc} x \quad (x > 0).$$

With  $x=1.72$ ,  $m=15$  we obtain

$\mu$	$w_\mu^{15}(1.72)$	$\mu$	$w_\mu^{15}(1.72)$	$\mu$	$w_\mu^{15}(1.72)$	$\mu$	$w_\mu^{15}(1.72)$
17	0	12	(3) 2.1011	7	(7) 2.5879	2	(11) 1.2920
16	1	11	(4) 1.3831	6	(8) 1.5569	1	(11) 6.0064
15	3.44	10	(4) 9.8005	5	(8) 8.9787	0	(12) 2.5820
14	(1) 4.3834	9	(5) 6.4143	4	(9) 4.9570	-1	(13) 1.0087
13	(2) 2.5399	8	(6) 4.1666	3	(10) 2.6031		

From **Table 7.1** we have  $\frac{2}{\sqrt{\pi}} e^{-(1.72)^2} = .058565$ .

Thus,

$$i \operatorname{erfc} 1.72 \approx (.058565)(6.0064 \times 10^{11}) / 1.0087 \times 10^{13} \\ = 3.4873 \times 10^{-3}$$

$$i^2 \operatorname{erfc} 1.72 \approx (.058565)(1.2920 \times 10^{11}) / 1.0087 \times 10^{13} \\ = 7.5013 \times 10^{-4}$$

$$i^3 \operatorname{erfc} 1.72 \approx (.058565)(2.6031 \times 10^{10}) / 1.0087 \times 10^{13} \\ = 1.5114 \times 10^{-4}.$$

**Example 6.** Compute  $C(8.65)$  using **Table 7.8**.

With  $x=8.65$ ,  $1/x=.115607$  we have from **Table 7.8** by linear interpolation

$$f(8.65) = .036797, \quad g(8.65) = .000159.$$

From **Table 4.6**

$$\sin\left(\frac{\pi}{2} x^2\right) = -.961382, \quad \cos\left(\frac{\pi}{2} x^2\right) = -.275218.$$

Using 7.3.9

$$C(8.65) = .5 + (.036797)(-.961382) \\ - (.000159)(-.275218) = .46467.$$

**Example 7.** Compute  $S_1(1.1)$  to 10D.

Using 7.3.8 and 7.3.10 we obtain by 6-pt interpolation in **Table 7.8**

$$S_1(1.1) = S\left(1.1 \sqrt{\frac{2}{\pi}}\right) \\ = S(.87767 \ 30169) = .31865 \ 57172.$$

**Example 8.** Compute  $S_2(5.24)$  to 6D.

Enter **Table 7.7** in the column headed by  $u$ . Using Aitken's scheme of interpolation

$u$	$S_2(u)$					
5.20310 58	.43280 06	.03689 42				
5.31808 80	.41573 97	-.07808 80	.42732 63			
5.08938 01	.45093 88	.15061 99	691 63	.42718 63		
5.43432 70	.39999 44	-.19432 70	756 60	6 52	.42717 71	
4.97691 11	.46990 94	.26308 89	674 79	9 39	61	.42717 67

$$S_2(5.24) = .427177$$

**Example 9.** Compute  $S_2(5.24)$  using Taylor's series and **Table 7.8**.

Using 7.3.21 we can write Taylor's series for  $f_2(u)$   $= f\left(\sqrt{\frac{2u}{\pi}}\right)$  and  $g_2(u) = g\left(\sqrt{\frac{2u}{\pi}}\right)$  in the form

$$f_2(u) = c_0 + c_1(u-u_0) + \frac{c_2}{2!}(u-u_0)^2 + \frac{c_3}{3!}(u-u_0)^3 + \dots,$$

$$g_2(u) = -\left[ c_1 + c_2(u-u_0) + \frac{c_3}{2!}(u-u_0)^2 + \frac{c_4}{3!}(u-u_0)^3 + \dots \right],$$

where

$$c_0 = f_2(u_0), c_1 = -g_2(u_0), \\ c_{k+2} = -c_k + (-1)^k \frac{1 \cdot 3 \dots (2k-1)}{\sqrt{2\pi} u_0 (2u_0)^k} \\ (k=0, 1, 2, \dots).$$

Consulting **Table 7.8** we chose  $u_0 = 1/.185638 = 5.386819$ , thus having  $u-u_0 = 5.24-5.386819 = -.146819$ . From **Table 7.8**

$$f_2(u_0) = .168270, g_2(u_0) = .014483.$$

Hence, applying the series above,

$$f_2(5.24) = .170436, g_2(5.24) = .015030.$$

Using the 4th formula at the bottom of **Table 7.8**

$$S_2(5.24) = .5 - (.170436)(.503471) \\ - (.015030)(-.864012) = .42718.$$

**Example 10.** Compute  $S_2(2)$  using **7.3.16**.

Generating the values of  $J_{n+1/2}(2)$  as described in chapter 10 we find

$$S_2(2) = J_{3/2}(2) + J_{7/2}(2) + J_{11/2}(2) + J_{15/2}(2) + \dots \\ = .49129 + .06852 + .00297 + .00006 = .56284.$$

**Example 11.** Compute  $\int_1^\infty \frac{Y_0(t)}{t} dt$  by numerical integration using **Tables 9.1** and **7.8**. [ $Y_0(t)$  is the Bessel function of the second kind defined in **9.1.16**.]

We decompose the integral into three parts

$$\int_1^\infty Y_0(t) \frac{dt}{t} = \int_1^{10} Y_0(t) \frac{dt}{t} + \int_{10}^\infty [Y_0(t) - \tilde{Y}_0(t)] \frac{dt}{t} \\ + \int_{10}^\infty \tilde{Y}_0(t) \frac{dt}{t}$$

where

$$\tilde{Y}_0(t) = \left(1 - \frac{9}{128t^2}\right) \frac{\sin\left(t - \frac{\pi}{4}\right)}{\sqrt{\frac{1}{2}\pi t}} \\ - \left(1 - \frac{75}{128t^2}\right) \frac{\cos\left(t - \frac{\pi}{4}\right)}{8t\sqrt{\frac{1}{2}\pi t}}$$

represents the first two terms of the asymptotic expansion **9.2.2**.

By numerical integration, using **Table 9.1**,

$$\int_1^{10} Y_0(t) \frac{dt}{t} = .41826 \ 00.$$

Using the fact that the remainder terms of the asymptotic expansion are less in absolute value than the first neglected terms, we can estimate

$$\left| \int_{10}^\infty [Y_0(t) - \tilde{Y}_0(t)] \frac{dt}{t} \right| \leq \sqrt{\frac{2}{\pi}} \int_{10}^\infty \left[ \frac{3^2 \cdot 5^2 \cdot 7^2}{2^{12} \cdot 4!} t^{-11/2} \right. \\ \left. + \frac{3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2}{2^{15} \cdot 5!} t^{-13/2} \right] dt = 7.33 \times 10^{-7}.$$

Finally,

$$\int_{10}^\infty \tilde{Y}_0(t) \frac{dt}{t} = \frac{14659}{6720} \sqrt{2} [1 - C_2(10) - S_2(10)] \\ - \frac{5953819 \cos 10 - \sin 10}{2688000 \sqrt{10\pi}} \\ - \frac{23107 \cos 10 + \sin 10}{2150400 \sqrt{10\pi}} = -.02298 \ 78,$$

using **Tables 7.8** and **4.8**. Hence

$$\int_1^\infty Y_0(t) \frac{dt}{t} = .41826 \ 00 - .02298 \ 78 = .39527 \ 22.$$

The answer correct to 8D is .39527 290 (**Table 11.2**).

**Example 12.** Compute  $w(.44 + .67i)$  using bivariate linear interpolation.

By linear interpolation in **Table 7.9** along the  $x$ -direction at  $y = .6$  and  $y = .7$

$$w(.44 + .6i) \approx .6(.522246 + .167880i) + .4(.498591 \\ + .202666i) = .512784 + .181794i$$

$$w(.44 + .7i) \approx .6(.487556 + .147975i) + .4(.467521 \\ + .179123i) = .479542 + .160434i.$$

By linear interpolation along the  $y$ -direction at  $x = .44$

$$w(.44 + .67i) \approx .3(.512784 + .181794i) + .7(.479542 \\ + .160434i) = .489515 + .166842i.$$

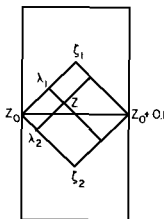
The correct answer is .489557 + .166889i.

**Example 13.** Compute  $\mathcal{R}w(z)$  for  $z = .44 + .61i$ . Bivariate linear interpolation, as described in **Example 12**, is most accurate if  $z$  lies near the center or along a diagonal of one of the squares of the tabular grid [7.6]. It is not as accurate for  $z$  near the midpoint of a side of a square, as in this example. However, we may introduce an auxil-

ary square (see diagram) which contains  $z$  close to its center. Bivariate linear interpolation can then be applied within this auxiliary square.

The values of  $w(z)$  needed at  $z=\zeta_1$ , and  $z=\zeta_2$  are easily approximated by the average of the four neighboring tabular values. Furthermore the parts to be used are given by

$$\frac{|z_0 - \lambda_1|}{|z_0 - \zeta_1|} = p_1 + p_2, \quad \frac{|z_0 - \lambda_2|}{|z_0 - \zeta_2|} = p_1 - p_2$$



where  $z = z_0 + .1(p_1 + ip_2)$ . Thus, with  $z_0 = .4 + .6i$ ,  $\zeta_1 = .45 + .65i$ ,  $\zeta_2 = .45 + .55i$ ,  $p_1 = .4$ ,  $p_2 = .1$ , we get from Table 7.9

$$\mathcal{R}w(\zeta_1) \approx \frac{1}{4}(.522246 + .498591 + .487556 + .467521) = .493979$$

$$\mathcal{R}w(\zeta_2) \approx \frac{1}{4}(.522246 + .498591 + .561252 + .533157) = .528812$$

$$\mathcal{R}w(z) \approx [1 - (.4 + .1)]\{[1 - (.4 - .1)].522246 + (.4 - .1).528812\} + (.4 + .1) \times \{[1 - (.4 - .1)] .493979 + (.4 - .1).498591\} = .509789.$$

The correct answer is .509756. Straightforward bivariate interpolation gives .509460.

**Example 14.** Compute  $\mathcal{J}w(.39 + .61i)$  to 6D using Taylor's series.

Let  $z = .39 + .61i$ ,  $z_0 = .4 + .6i$ . From 7.1.20, and using Table 7.9, we have

$$w(z_0) = .522246 + .167880i$$

$$w'(z_0) = -.21634 + .36738i, \quad z - z_0 = (-1 + i)10^{-2}$$

$$\frac{1}{2}w''(z_0) = -.215 - .185i, \quad (z - z_0)^2 = -2i \times 10^{-4}$$

$$\mathcal{J}w(z) = .167880 - .0021634 - .0036738 + .0000430 = .162086.$$

**Example 15.** Compute  $w(.4 - 1.3i)$ .

From 7.1.11, 7.1.12

$$w(.4 - 1.3i) = \overline{w(-.4 - 1.3i)} = 2e^{-(.4 - 1.3i)^2} - \overline{w(.4 + 1.3i)}.$$

Using Tables 7.9, 4.4 and 4.6

$$w(.4 - 1.3i) = 4.33342 + 8.04201i.$$

**Example 16.** Compute  $w(7 + 2i)$ .

Using the second formula at the end of Table 7.9

$$w(7 + 2i) = (-2 + 7i) \left( \frac{.5124242}{44.72474 + 28i} + \frac{.05176536}{42.27525 + 28i} \right) = .021853 + .075010i.$$

**Example 17.** Compute  $\text{erf}(2 + i)$ .

From 7.1.3, 7.1.12 we have

$$\text{erf } z = 1 - e^{-z^2} w(iz) = 1 - e^{y^2 - z^2} (\cos 2xy - i \sin 2xy) \overline{w(y + ix)} \quad (z = x + iy).$$

Using Tables 7.9, 4.4, 4.6

$$\text{erf}(2 + i) = 1 - e^{-3} (\cos 4 - i \sin 4) \overline{w(1 + 2i)} = 1.003606 - .0112590i.$$

**Example 18.** Compute  $S_1\left(\left(\frac{1}{2} + i\right)\sqrt{2}\right)$ .

From 7.3.22, 7.3.8, 7.3.18 we have

$$S_1(z) = \frac{1}{2} - \frac{1-i}{4} e^{iz^2} w\left[(1+i)\frac{z}{\sqrt{2}}\right] - \frac{1+i}{4} e^{-iz^2} w\left[(i-1)\frac{z}{\sqrt{2}}\right].$$

Setting  $z = \left(\frac{1}{2} + i\right)\sqrt{2}$  and making use of 7.1.11, 7.1.12, and Table 7.9

$$S_1\left(\left(\frac{1}{2} + i\right)\sqrt{2}\right) = -\frac{i}{2} - \frac{1-i}{4} e^{-2} \left(\cos \frac{3}{2} - i \sin \frac{3}{2}\right) \overline{w\left(\frac{1}{2} + \frac{3}{2}i\right)} + \frac{1+i}{4} e^2 \left(\cos \frac{3}{2} + i \sin \frac{3}{2}\right) w\left(\frac{3}{2} + \frac{1}{2}i\right) = -.990734 - .681619i.$$

**Example 19.** Compute  $\int_0^\infty e^{-(1/4)t^2 - 3t} \cos(2t) dt$  using Table 7.9.

Setting  $b = y + ix$ ,  $c = 0$  in 7.4.2 and using 7.1.3, 7.1.12 we find

$$\int_0^\infty e^{-at^2 - 2yt} \cos(2xt) dt = \frac{1}{2} \sqrt{\frac{\pi}{a}} \mathcal{R}w\left(\frac{x + iy}{\sqrt{a}}\right) \quad (a > 0, x, y \text{ real}).$$

Hence from Table 7.9

$$\int_0^\infty e^{-(1/4)t^2 - 3t} \cos(2t) dt = \sqrt{\pi} \mathcal{R}w(2 + 3i) = .231761.$$

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- $\rho=\rho_\theta(.001)\rho'_\theta(.01)\rho''_\theta(.0002)5$ ,  $0 \leq \rho_\theta < \rho'_\theta \leq \rho''_\theta \leq 5$ , 5D;
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